

Analytic Continuation of Multiple Zeta-Functions and the Asymptotic Behavior at Non-Positive Integers

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1 Introduction

The Euler-Zagier multiple zeta function $\zeta_d(s_1, \dots, s_d)$ is defined by

$$\zeta_d(s_1, \dots, s_d) := \sum_{m_1=1}^{\infty} \cdots \sum_{m_d=1}^{\infty} \frac{1}{m_1^{s_1} (m_1 + m_2)^{s_2} \cdots (m_1 + \cdots + m_d)^{s_d}} \quad (1.1)$$

where s_i ($i = 1, \dots, d$) are complex variables. Matsumoto [6] proved that the series (1.1) is absolutely convergent in

$$\left\{ (s_1, \dots, s_d) \in \mathbb{C}^d \mid \Re(s_d(d-k+1)) > k \ (k = 1, \dots, d) \right\}$$

where $s_d(n) = s_n + s_{n+1} + \cdots + s_d$ ($n = 1, \dots, d$). Akiyama, Egami and Tanigawa [1] and Zhao [10] proved the meromorphic continuation to the whole space independently. Akiyama, Egami and Tanigawa used the Euler-Maclaurin summation formula and Zhao used generalized functions to prove the analytic continuation. Later, Matsumoto [7] also proved the analytic continuation using Mellin-Barnes integral formula.

The function $\zeta_d(s_1, \dots, s_d)$ has singularities on

$$\begin{cases} s_d = 1, \\ s_{d-1} + s_d = 2, 1, 0, -2, -4, \dots, \\ s_d(d-j+1) \in \mathbb{Z}_{\leq j} \ (j = 3, 4, \dots, d), \end{cases} \quad (1.2)$$

where $\mathbb{Z}_{\leq j}$ is the set of integers less than or equal to j ; $\mathbb{Z}_{\geq j}$ is defined similarly. Therefore $(-r_1, \dots, -r_d) \in \mathbb{Z}_{\leq 0}^d$ lies on the set of singularities. Moreover, it is an indeterminacy of $\zeta_d(s_1, \dots, s_d)$. For example, Sasaki [8] proved that

$$\lim_{s_3 \rightarrow 0} \lim_{s_2 \rightarrow 0} \lim_{s_1 \rightarrow 0} \zeta_3(s_1, s_2, s_3) = -\frac{3}{8}, \quad (1.3)$$

$$\lim_{s_1 \rightarrow 0} \lim_{s_2 \rightarrow 0} \lim_{s_3 \rightarrow 0} \zeta_3(s_1, s_2, s_3) = -\frac{1}{4}. \quad (1.4)$$

Since $(0, 0, 0)$ is an indeterminacy of $\zeta_3(s_1, s_2, s_3)$, (1.3) and (1.4) give different values.

Akiyama, Egami and Tanigawa [1] defined the regular values by

$$\zeta_d(-r_1, \dots, -r_d) := \lim_{s_1 \rightarrow -r_1} \cdots \lim_{s_d \rightarrow -r_d} \zeta_d(s_1, \dots, s_d),$$

and Akiyama and Tanigawa [2] considered the reverse and central values given by

$$\begin{aligned} \zeta_d^R(-r_1, \dots, -r_d) &:= \lim_{s_d \rightarrow -r_d} \cdots \lim_{s_1 \rightarrow -r_1} \zeta_d(s_1, \dots, s_d), \\ \zeta_d^C(-r_1, \dots, -r_d) &:= \lim_{\varepsilon \rightarrow 0} \zeta_d(-r_1 + \varepsilon, \dots, -r_d + \varepsilon), \end{aligned}$$

respectively. Further, Sasaki [8] generalized the regular and reverse values. He defined multiple zeta values for coordinatewise limits by

$$\zeta_d(-r_1, \dots, -r_d) := \lim_{\substack{s_j \rightarrow -r_j \\ i_j=d}} \cdots \lim_{\substack{s_j \rightarrow -r_j \\ i_j=1}} \zeta_d(s_1, \dots, s_d),$$

where $\{i_1, \dots, i_d\} = \{1, \dots, d\}$. He obtained all multiple zeta values of depth 3 for coordinatewise limits. In addition, he treated the multiple zeta values of depth 4 for coordinatewise limits in [9]. On the other hand, Kamano [4] considered the regular, reverse and central values of the multiple Hurwitz zeta functions. Komori [5] considered more general multiple zeta functions, and he obtained multiple zeta values at non-positive integers given by

$$\begin{aligned} \zeta_d(-\mathbf{r})^w &= \lim_{z_{w-1(d)} \rightarrow -r_{w-1(d)}} \cdots \lim_{z_{w-1(1)} \rightarrow -r_{w-1(1)}} \zeta_d(z_1, \dots, z_d), \\ \zeta_d(-\mathbf{r})_{\boldsymbol{\theta}} &= \zeta_d(-r_1, \dots, -r_d)_{\boldsymbol{\theta}} = \lim_{\delta \rightarrow 0} \zeta_d(-r_1 + \delta\theta_1, \dots, -r_d + \delta\theta_d), \end{aligned}$$

where $-\mathbf{r} = (-r_1, \dots, -r_d) \in \mathbb{Z}_{\leq 0}^d$, $w \in \mathfrak{S}_d$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d) \in \mathbb{C}^d$. To obtain these values by Komori's method, we need to compute generalized multiple Bernoulli numbers.

In the present paper, we prove two theorems. Theorem 1 gives the meromorphic continuation of the multiple zeta function to the whole space. The meromorphic continuation was already proved. Proof of Theorem 1 is similar to the proof of meromorphic continuation in [10]. In [10], Zhao used the theory of generalized functions [3] to prove the meromorphic continuation. On the other hand, to prove Theorem 1, we do not use the theory of generalized functions but integration by parts. In Theorem 2, we prove asymptotic behavior near the non-positive integers. Until now, we have been able to get only 2 kinds of the limit values, $\zeta(-\mathbf{r})$ and $\zeta_d(-\mathbf{r})_{\boldsymbol{\theta}}^w$. Using Theorem 2, we can compute

not only $\zeta(-\mathbf{r})$, $\zeta_d(-\mathbf{r})_{\boldsymbol{\theta}}^w$ but also various different types of limit values. In fact, by Theorem 2, we can compute, for example,

$$\lim_{\varepsilon \rightarrow 0} \zeta_3(\varepsilon^2, \varepsilon, \varepsilon) = -\frac{1}{3}. \quad (1.5)$$

This limit value is not contained in the above 2 kinds of values, however by Theorem 2, we can compute this value.

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2 Main theorems

In this section, we state two theorems.

Let B_m be the m th Bernoulli number, and $B(x, y)$ be the beta function. For $(m_1, \dots, m_d) \in \mathbb{Z}_{\geq 0}^d$, $(p_1, \dots, p_d) \in \mathbb{Z}_{\geq 0}^d$ and $(\varepsilon_1, \dots, \varepsilon_d) \in \mathbb{C}^d$, let $m_d(n)$, $p_d(n)$ and $\varepsilon_d(n)$ be $m_n + m_{n+1} + \dots + m_d$, $p_n + p_{n+1} + \dots + p_d$ and $\varepsilon_n + \varepsilon_{n+1} + \dots + \varepsilon_d$ respectively. In addition, Pochhammer symbol $(a)_n$ is defined by $(a)_n := \Gamma(a+n)/\Gamma(a)$. In this paper, symmetric group \mathfrak{S} is defined by $\{\sigma | \sigma : \{2, \dots, d\} \rightarrow \{2, \dots, d\}, \sigma \text{ is a bijective function}\}$.

Theorem 1. For $d \geq 2$ and $n_1, \dots, n_d \in \mathbb{Z}_{\geq 0}$, $\zeta_d(s_1, \dots, s_d)$ can be continued meromorphically to

$$\left\{ (s_1, \dots, s_d) \in \mathbb{C}^d \mid \Re(s_d(j)) > d - j - n_j \ (j = 1, \dots, d), \ \Re(s_{j-1}) > -n_j - 1 \ (j = 2, \dots, d) \right\},$$

and $\zeta_d(s_1, \dots, s_d)$ can be represented by

$$\begin{aligned} & \zeta_d(s_1, \dots, s_d) \\ &= \frac{1}{\Gamma(s_1) \cdots \Gamma(s_d)} \sum_{k=0}^{n_1} \sum_{p_1+\dots+p_d=k} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} \frac{1}{s_d(1) - d + k} \prod_{j=2}^d B(s_d(j) - d + j + p_d(j) - 1, s_{j-1}) \\ &+ \frac{1}{\Gamma(s_1) \cdots \Gamma(s_d)} \int_0^1 x_1^{s_d(1)-d+n_1} F_{\varphi_a}(x_1) dx_1 \\ &+ \frac{1}{\Gamma(s_1) \cdots \Gamma(s_d)} \int_1^\infty \frac{x_1^{s_d(1)-d}}{e^{x_1} - 1} F_{\psi_a}(x_1) dx_1, \end{aligned} \quad (2.1)$$

where

$$\begin{aligned} & F_{f_a}(x_1) \\ &:= \sum_{(a_2, \dots, a_d)} \sum_{m=1}^d \sum_{\substack{\sigma(2) < \dots < \sigma(m) \\ \sigma(m+1) < \dots < \sigma(d)}} \sum_{k_{\sigma(2)}=0}^{n_{\sigma(2)}} \cdots \sum_{k_{\sigma(m)}=0}^{n_{\sigma(m)}} \left\{ \prod_{j=2}^m (-1)^{k_{\sigma(j)}} (u_{\sigma(j)} + 1)^{-1}_{k_{\sigma(j)}+1} \left(\frac{1}{2} \right)^{u_{\sigma(j)} + k_{\sigma(j)} + 1} \right\} \\ &\times \left\{ \prod_{j=m+1}^d (-1)^{n_{\sigma(j)}+1} (u_{\sigma(j)} + 1)^{-1}_{n_{\sigma(j)}+1} \right\} \int_0^{\frac{1}{2}} \cdots \int_0^{\frac{1}{2}} \left(\prod_{j=m+1}^d x_{\sigma(j)}^{u_{\sigma(j)} + n_{\sigma(j)} + 1} \right) \\ &\left[\frac{d^{k_{\sigma(2)}}}{dx_{\sigma(2)}^{k_{\sigma(2)}}} \cdots \frac{d^{k_{\sigma(m)}}}{dx_{\sigma(m)}^{k_{\sigma(m)}}} \frac{d^{n_{\sigma(m+1)}+1}}{dx_{\sigma(m+1)}^{n_{\sigma(m+1)}+1}} \cdots \frac{d^{n_{\sigma(d)}+1}}{dx_{\sigma(d)}^{n_{\sigma(d)}+1}} f_a(x_1, \dots, x_d) \right]_{\substack{x_{\sigma(2)}=\frac{1}{2} \\ \vdots \\ x_{\sigma(m)}=\frac{1}{2}}} dx_{\sigma(m+1)} \cdots dx_{\sigma(d)}, \\ &\varphi_a(x_1, \dots, x_d) := \left(\prod_{j=2}^d (1 - x_j)^{v_j} \right) \sum_{k=n_1+1}^\infty \sum_{p_1+\dots+p_d=k} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} x_1^{k-n_1-1} t_2^{p_d(2)} \cdots t_d^{p_d(d)}, \\ &\psi_a(x_1, \dots, x_d) := \prod_{j=2}^d (1 - x_j)^{v_j} \frac{x_1 t_2 \cdots t_j}{e^{x_1 t_2 \cdots t_j} - 1}. \end{aligned}$$

Here, the summation $\sum_{(a_2, \dots, a_d)}$ runs all combinations of $a_j = 0$ or 1 ($j = 2, \dots, d$), and $\sum_{\substack{\sigma(2) < \dots < \sigma(m) \\ \sigma(m+1) < \dots < \sigma(d)}}$ runs all $\sigma \in \mathfrak{S}$ satisfying $\sigma(2) < \dots < \sigma(m)$ and $\sigma(m+1) < \dots < \sigma(d)$, and u_j , v_j and t_j are defined by

$$\begin{aligned} u_j &:= \begin{cases} s_d(j) - d + j - 2 & (a_j = 0), \\ s_{j-1} - 1 & (a_j = 1), \end{cases} & v_j &:= \begin{cases} s_{j-1} - 1 & (a_j = 0), \\ s_d(j) - d + j - 2 & (a_j = 1), \end{cases} \\ t_j &:= \begin{cases} x_j & (a_j = 0), \\ 1 - x_j & (a_j = 1). \end{cases} \end{aligned}$$

The function $\zeta_d(s_1, \dots, s_d)$ has possible singularities on

$$\left\{ (s_1, \dots, s_d) \in \mathbb{C}^d \mid s_d(j) \in \mathbb{Z}_{\leq d-j+1}, s_j \in \mathbb{Z}_{\leq 0} \ (j = 1, \dots, d) \right\}.$$

Using Theorem 1, we can obtain the following Theorem 2.

Theorem 2. Suppose that $\varepsilon_j \neq 0$, $\varepsilon_d(j) \neq 0$ ($j = 1, \dots, d$), $|\varepsilon_1| + \dots + |\varepsilon_d| \leq \frac{1}{2}$ and $|\varepsilon_k/\varepsilon_d(j)| \ll 1$ as $(\varepsilon_1, \dots, \varepsilon_d) \rightarrow (0, \dots, 0)$ ($j = 1, \dots, d$, $k = j, \dots, d$). Then for $m_j \in \mathbb{Z}_{\geq 0}$ ($j = 1, \dots, d$), we have

$$\begin{aligned}
& \zeta_d(-m_1 + \varepsilon_1, \dots, -m_d + \varepsilon_d) = (-1)^{m_d} m_d! \\
& \times \sum_{p_1 + \dots + p_d = d+M} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} \prod_{j=2}^d h(-m_d(j) - d + j + p_d(j) - 1, -m_d(j-1) - d + j + p_d(j) - 1) \\
& \times \frac{[\varepsilon_d(j)]_{-m_d(j)-d+j+p_d(j)-1}}{[\varepsilon_d(j-1)]_{-m_d(j-1)-d+j+p_d(j)-1}} + \sum_{j=1}^d O(\varepsilon_j) \\
& = (-1)^{m_d} m_d! \sum_{\substack{p_1 + \dots + p_d = d+M \\ -m_d(j)-d+j+p_d(j) < 2 \text{ or} \\ -m_d(j-1)-d+j+p_d(j) \geq 2 \ (2 \leq j \leq d)}} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} \prod_{j=2}^d \frac{[\varepsilon_d(j)]_{-m_d(j)-d+j+p_d(j)-1}}{[\varepsilon_d(j-1)]_{-m_d(j-1)-d+j+p_d(j)-1}} + \sum_{j=1}^d O(\varepsilon_j)
\end{aligned}$$

as $(\varepsilon_1, \dots, \varepsilon_d) \rightarrow (0, \dots, 0)$, where

$$\begin{aligned}
M &:= m_1 + \dots + m_d, \\
[a]_n &:= \begin{cases} a(n-1)! & (n \geq 1), \\ (-1)^n (-n)!^{-1} & (n < 0), \end{cases} \\
h(m, n) &:= \begin{cases} 0 & (m \geq 1 > n), \\ 1 & (\text{otherwise}). \end{cases}
\end{aligned}$$

In Theorem 2, ε_j ($j = 1, \dots, d$) should satisfy $|\varepsilon_k/\varepsilon_d(j)| \ll 1$ ($j = 1, \dots, d$, $k = j, \dots, d$). Let us think about this condition. If $|\varepsilon_k/\varepsilon_d(j)| \rightarrow \infty$, then $\varepsilon_d(j)$ tends to 0 rapidly. By (1.2), $s_j + \dots + s_d = -M$ is a singular locus. Therefore, when $|\varepsilon_k/\varepsilon_d(j)| \rightarrow \infty$, the point $(-m_1 + \varepsilon_1, \dots, -m_d + \varepsilon_d)$ approximates asymptotically to a singular locus. Hence, $|\varepsilon_k/\varepsilon_d(j)| \ll 1$ means geometrically that $(-m_1 + \varepsilon_1, \dots, -m_d + \varepsilon_d)$ does not approximate asymptotically to a singular locus.

3 Examples

By Theorem 2, we can compute various multiple zeta values at non-positive integers. Let us see some examples.

In the case $d = 2$, we have

$$\begin{aligned}
\zeta_2(\varepsilon_1, \varepsilon_2) &= \frac{1}{3} + \frac{1}{24} \cdot \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} + \sum_{j=1}^2 O(\varepsilon_j), \\
\zeta_2(-1 + \varepsilon_1, \varepsilon_2) &= \frac{1}{24} + \sum_{j=1}^2 O(\varepsilon_j), \\
\zeta_2(\varepsilon_1, -1 + \varepsilon_2) &= \frac{1}{12} + \sum_{j=1}^2 O(\varepsilon_j), \\
\zeta_2(-1 + \varepsilon_1, -1 + \varepsilon_2) &= \frac{1}{360} + \frac{1}{720} \cdot \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} + \sum_{j=1}^2 O(\varepsilon_j).
\end{aligned}$$

In the case $d = 3$, we have

$$\begin{aligned}\zeta_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) &= -\frac{1}{4} - \frac{1}{24} \cdot \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} - \frac{1}{24} \cdot \frac{\varepsilon_2 + 2\varepsilon_3}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} + \sum_{j=1}^3 O(\varepsilon_j), \\ \zeta_3(-1 + \varepsilon_1, \varepsilon_2, \varepsilon_3) &= -\frac{17}{720} - \frac{1}{144} \cdot \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} + \frac{1}{720} \cdot \frac{-\varepsilon_2 + 3\varepsilon_3}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} + \sum_{j=1}^3 O(\varepsilon_j), \\ \zeta_3(\varepsilon_1, -1 + \varepsilon_2, \varepsilon_3) &= -\frac{19}{360} + \frac{1}{360} \cdot \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} + \sum_{j=1}^3 O(\varepsilon_j), \\ \zeta_3(\varepsilon_1, \varepsilon_2, -1 + \varepsilon_3) &= -\frac{3}{40} - \frac{1}{720} \cdot \frac{4\varepsilon_2 + 3\varepsilon_3}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} + \sum_{j=1}^3 O(\varepsilon_j).\end{aligned}$$

Note that the example (1.5) comes from the first example of the above, taking $\varepsilon_1 = \varepsilon^2$ and $\varepsilon_2 = \varepsilon_3 = \varepsilon$.

In the case $d = 4$, we have

$$\begin{aligned}\zeta_4(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) &= \frac{1}{5} + \frac{1}{36} \cdot \frac{\varepsilon_4}{\varepsilon_3 + \varepsilon_4} + \frac{1}{48} \cdot \frac{\varepsilon_3 + 2\varepsilon_4}{\varepsilon_2 + \varepsilon_3 + \varepsilon_4} + \frac{1}{720} \cdot \frac{19\varepsilon_2 + 33\varepsilon_3 + 52\varepsilon_4}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4} \\ &\quad + \frac{1}{144} \cdot \frac{\varepsilon_4(\varepsilon_2 + \varepsilon_3 + \varepsilon_4)}{(\varepsilon_3 + \varepsilon_4)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)} + \sum_{j=1}^4 O(\varepsilon_j).\end{aligned}$$

4 Lemmas

To prove Theorem 1 and Theorem 2, in this section, we prove several lemmas. In this section, suppose that $\Re(s_k) > 1$ ($k = 1, \dots, d$).

Lemma 1. *Let I_1 be an interval on \mathbb{R} and $f(x_1, \dots, x_d)$ be of class C^∞ on $I_1 \times [0, \frac{1}{2}] \times \dots \times [0, \frac{1}{2}] \subset \mathbb{R}^d$. Then for all (a_2, \dots, a_d) , $x_1 \in I_1$ and every $k = 2, \dots, d$, we have*

$$\begin{aligned}& \int_0^{\frac{1}{2}} \cdots \int_0^{\frac{1}{2}} \left(\prod_{j=2}^d x_j^{u_j} \right) f(x_1, \dots, x_d) dx_2 \cdots dx_d \\ &= \sum_{m=1}^k \sum_{\substack{\sigma(k;2) < \dots < \sigma(k;m) \\ \sigma(k;m+1) < \dots < \sigma(k;k)}} \left\{ \prod_{j=2}^m (u_{\sigma(k;j)} + 1)^{-1} \left(\frac{1}{2} \right)^{u_{\sigma(k;j)} + 1} \right\} \left\{ \prod_{j=m+1}^k (-1)(u_{\sigma(k;j)} + 1)^{-1} \right\} \\ &\quad \times \int_0^{\frac{1}{2}} \cdots \int_0^{\frac{1}{2}} \left(\prod_{j=m+1}^k x_{\sigma(k;j)}^{u_{\sigma(k;j)} + 1} \right) \left(\prod_{j=k+1}^d x_j^{u_j} \right) \\ &\quad \left[\frac{d}{dx_{\sigma(k;m+1)}} \cdots \frac{d}{dx_{\sigma(k;k)}} f(x_1, \dots, x_d) \right]_{\substack{x_{\sigma(k;2)} = \frac{1}{2} \\ \vdots \\ x_{\sigma(k;m)} = \frac{1}{2}}} dx_{\sigma(k;m+1)} \cdots dx_{\sigma(k;k)} dx_{k+1} \cdots dx_d, \tag{4.1}\end{aligned}$$

where $\sigma(k; \cdot)$ is an element of the group given by

$$\mathfrak{S}_k = \{ \sigma(k; \cdot) \mid \sigma(k; \cdot) : \{2, \dots, k\} \rightarrow \{2, \dots, k\}, \sigma(k; \cdot) \text{ is a bijective function} \}.$$

Proof. In the case of $k = 2$, using integration by parts with respect to x_2 on the left-hand side of (4.1), we have

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \cdots \int_0^{\frac{1}{2}} \left(\prod_{j=2}^d x_j^{u_j} \right) f(x_1, \dots, x_d) dx_2 \cdots dx_d \\
&= (u_2 + 1)^{-1} \left(\frac{1}{2} \right)^{u_2+1} \int_0^{\frac{1}{2}} \cdots \int_0^{\frac{1}{2}} \left(\prod_{j=3}^d x_j^{u_j} \right) [f(x_1, \dots, x_d)]_{x_2=\frac{1}{2}} dx_3 \cdots dx_d \\
&\quad - (u_2 + 1)^{-1} \int_0^{\frac{1}{2}} \cdots \int_0^{\frac{1}{2}} x_2^{u_2+1} \left(\prod_{j=3}^d x_j^{u_j} \right) \left[\frac{\partial}{\partial x_2} f(x_1, \dots, x_d) \right] dx_2 \cdots dx_d.
\end{aligned} \tag{4.2}$$

The first term on the right-hand side of (4.2) is the term corresponding to $m = 2$ of (4.1), and the second term is the term corresponding to $m = 1$ of (4.1).

Suppose that Lemma 1 holds for $k - 1$. Using integration by parts with respect to x_k on the right-hand side of (4.1), we have

$$\begin{aligned}
& (u_k + 1)^{-1} \left(\frac{1}{2} \right)^{u_k+1} \left[\frac{d}{dx_{\sigma(k-1;m+1)}} \cdots \frac{d}{dx_{\sigma(k-1;k-1)}} f(x_1, \dots, x_d) \right]_{\substack{x_{\sigma(k-1;2)}=\frac{1}{2} \\ \vdots \\ x_{\sigma(k-1;m)}=\frac{1}{2} \\ x_k=\frac{1}{2}}} \\
& - (u_k + 1)^{-1} \int_0^{\frac{1}{2}} x_k^{u_k+1} \left[\frac{\partial}{\partial x_k} \frac{d}{dx_{\sigma(k-1;m+1)}} \cdots \frac{d}{dx_{\sigma(k-1;k-1)}} f(x_1, \dots, x_d) \right]_{\substack{x_{\sigma(k-1;2)}=\frac{1}{2} \\ \vdots \\ x_{\sigma(k-1;m)}=\frac{1}{2}}} dx_k.
\end{aligned} \tag{4.3}$$

The first term of (4.3) is the term corresponding to $k = \sigma(k; m)$ of (4.1), and the second term of (4.3) is the term corresponding to $k = \sigma(k; k)$ of (4.1). \square

Lemma 2. Let $f(x_1, \dots, x_d)$ be as in Lemma 1. Then for all (a_2, \dots, a_d) , $x_1 \in I_1$, we have

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \cdots \int_0^{\frac{1}{2}} \left(\prod_{j=2}^d x_j^{u_j} \right) f(x_1, \dots, x_d) dx_2 \cdots dx_d \\
&= \sum_{m=1}^d \sum_{\substack{\sigma(2) < \cdots < \sigma(m) \\ \sigma(m+1) < \cdots < \sigma(d)}} \sum_{k_{\sigma(2)}=0}^{n_{\sigma(2)}} \cdots \sum_{k_{\sigma(m)}=0}^{n_{\sigma(m)}} \left\{ \prod_{j=2}^m (-1)^{k_{\sigma(j)}} (u_{\sigma(j)} + 1)_{k_{\sigma(j)}+1}^{-1} \left(\frac{1}{2} \right)^{u_{\sigma(j)} + k_{\sigma(j)} + 1} \right\} \\
& \times \left\{ \prod_{j=m+1}^d (-1)^{n_{\sigma(j)}+1} (u_{\sigma(j)} + 1)_{n_{\sigma(j)}+1}^{-1} \right\} \\
& \times \int_0^{\frac{1}{2}} \cdots \int_0^{\frac{1}{2}} \left(\prod_{j=m+1}^d x_{\sigma(j)}^{u_{\sigma(j)} + n_{\sigma(j)} + 1} \right) \\
& \left[\frac{d^{k_{\sigma(2)}}}{dx_{\sigma(2)}^{k_{\sigma(2)}}} \cdots \frac{d^{k_{\sigma(m)}}}{dx_{\sigma(m)}^{k_{\sigma(m)}}} \frac{d^{n_{\sigma(m+1)}+1}}{dx_{\sigma(m+1)}^{n_{\sigma(m+1)}+1}} \cdots \frac{d^{n_{\sigma(d)}+1}}{dx_{\sigma(d)}^{n_{\sigma(d)}+1}} f(x_1, \dots, x_d) \right]_{\substack{x_{\sigma(2)}=\frac{1}{2} \\ \vdots \\ x_{\sigma(m)}=\frac{1}{2}}} dx_{\sigma(m+1)} \cdots dx_{\sigma(d)},
\end{aligned} \tag{4.4}$$

where $n_2, \dots, n_d \in \mathbb{Z}_{\geq 0}$, σ are as in Theorem 1.

Proof. Induction on $n_2 + \cdots + n_d$.

In the case $n_2 = \cdots = n_d = 0$, (4.4) is valid by Lemma 1.

Suppose that (4.4) holds for (n_2, \dots, n_d) , and let us prove (4.4) for $(n_2, \dots, n_k + 1, \dots, n_d)$. The right-hand side of (4.4) is divided into two terms,

$$\begin{aligned}
& \sum_{m=1}^d \sum_{\substack{\sigma(2) < \cdots < \sigma(m) \\ \sigma(m+1) < \cdots < \sigma(d) \\ k \in \{\sigma(2), \dots, \sigma(m)\}}} \sum_{k_{\sigma(2)}=0}^{n_{\sigma(2)}} \cdots \sum_{k_{\sigma(m)}=0}^{n_{\sigma(m)}} + \sum_{m=1}^d \sum_{\substack{\sigma(2) < \cdots < \sigma(m) \\ \sigma(m+1) < \cdots < \sigma(d) \\ k \in \{\sigma(m+1), \dots, \sigma(d)\}}} \sum_{k_{\sigma(2)}=0}^{n_{\sigma(2)}} \cdots \sum_{k_{\sigma(m)}=0}^{n_{\sigma(m)}}.
\end{aligned} \tag{4.5}$$

The first term of (4.5) has no integral of x_k , and the second term of (4.5) has an integral of x_k .

Using integration by parts with respect to x_k on the second term of (4.5), we have

$$\begin{aligned}
& (u_k + n_k + 2)^{-1} \left\{ \left(\frac{1}{2} \right)^{u_k + n_k + 2} \left[\frac{d^{k_{\sigma(2)}}}{dx_{\sigma(2)}^{k_{\sigma(2)}}} \cdots \frac{d^{k_{\sigma(m)}}}{dx_{\sigma(m)}^{k_{\sigma(m)}}} \frac{d^{n_{\sigma(m+1)}+1}}{dx_{\sigma(m+1)}^{n_{\sigma(m+1)}+1}} \cdots \frac{d^{n_{\sigma(d)}+1}}{dx_{\sigma(d)}^{n_{\sigma(d)}+1}} f(x_1, \dots, x_d) \right]_{\substack{x_{\sigma(2)}=\frac{1}{2} \\ \vdots \\ x_{\sigma(m)}=\frac{1}{2} \\ x_k=\frac{1}{2}}} \right. \\
& \left. - \int_0^{\frac{1}{2}} x_k^{u_k + n_k + 2} \left[\frac{d^{k_{\sigma(2)}}}{dx_{\sigma(2)}^{k_{\sigma(2)}}} \cdots \frac{d^{k_{\sigma(m)}}}{dx_{\sigma(m)}^{k_{\sigma(m)}}} \frac{d^{n_{\sigma(m+1)}+1}}{dx_{\sigma(m+1)}^{n_{\sigma(m+1)}+1}} \cdots \frac{d^{n_k+2}}{dx_k^{n_k+2}} \cdots \frac{d^{n_{\sigma(d)}+1}}{dx_{\sigma(d)}^{n_{\sigma(d)}+1}} f(x_1, \dots, x_d) \right]_{\substack{x_{\sigma(2)}=\frac{1}{2} \\ \vdots \\ x_{\sigma(m)}=\frac{1}{2}}} dx_k \right\}.
\end{aligned} \tag{4.6}$$

Using (4.5) and (4.6), we find (4.4) for $(n_2, \dots, n_k + 1, \dots, n_d)$. \square

Lemma 3. $\varphi_a(x_1, \dots, x_d)$ is C^∞ on $[0, 1] \times [0, \frac{1}{2}] \times \dots \times [0, \frac{1}{2}] \subset \mathbb{R}^d$.

Proof. Since $(1 - x_j)^{v_j}$ ($j = 2, \dots, d$) are C^∞ on $[0, 1] \times [0, \frac{1}{2}] \times \dots \times [0, \frac{1}{2}]$, what we have to prove is that

$$\sum_{k=n_1+1}^{\infty} \sum_{p_1+\dots+p_d=k} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} x_1^{k-n_1-1} t_2^{p_d(2)} \cdots t_d^{p_d(d)}$$

is C^∞ on $[0, 1] \times [0, \frac{1}{2}] \times \dots \times [0, \frac{1}{2}]$. Clearly, we have

$$\begin{aligned} & \sum_{k=n_1+1}^{\infty} \sum_{p_1+\dots+p_d=k} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} x_1^{k-n_1-1} t_2^{p_d(2)} \cdots t_d^{p_d(d)} \\ &= \frac{\prod_{j=1}^d \frac{x_1 t_2 \cdots t_j}{e^{x_1 t_2 \cdots t_j} - 1} - \sum_{k=0}^{n_1} \sum_{p_1+\dots+p_d=k} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} x_1^k t_2^{p_d(2)} \cdots t_d^{p_d(d)}}{x_1^{n_1+1}}. \end{aligned} \quad (4.7)$$

We prove that the right-hand side of (4.7) is C^∞ . The numerator of the right-hand side is C^∞ , so the right-hand side is C^∞ except for $x_1 = 0$. We can find that $x_1 = 0$ is a removable singularity by observing the left-hand side of (4.7). Hence, $\varphi_a(x_1, \dots, x_d)$ is C^∞ on $[0, 1] \times [0, \frac{1}{2}] \times \dots \times [0, \frac{1}{2}]$. \square

In Lemma 4 and Lemma 5, we use

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d$.

Lemma 4. Let $\alpha = (0, \alpha_2, \dots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d$. Then we have

$$\partial^\alpha \left(\prod_{j=2}^d \frac{x_1 t_2 \cdots t_j}{e^{x_1 t_2 \cdots t_j} - 1} \right) = \sum_m \prod_{j=2}^d \frac{f_{m,j}(e^{x_1 t_2, \dots, t_j}, x_1, t_2, \dots, t_j)}{(e^{x_1 t_2 \cdots t_j} - 1)^{\alpha_2 + \dots + \alpha_j + 1}}$$

where \sum_m is a finite summation, $f_{m,j}$ is a polynomial which degree of $e^{x_1 t_2 \cdots t_j}$ is less than or equal to $\alpha_2 + \dots + \alpha_j$.

Proof. Induction on $|\alpha| = \alpha_1 + \dots + \alpha_d$.

When $|\alpha| = 0$, Lemma 4 is trivial.

Suppose that Lemma 4 is valid for $(0, \alpha_2, \dots, \alpha_d)$. Differentiating with respect to x_k , we have

$$\begin{aligned} & \frac{\partial}{\partial x_k} \partial^\alpha \left(\prod_{j=2}^d \frac{x_1 t_2 \cdots t_j}{e^{x_1 t_2 \cdots t_j} - 1} \right) \\ &= \sum_m \frac{\partial}{\partial x_k} \left(\prod_{j=2}^d \frac{f_{m,j}(e^{x_1 t_2, \dots, t_j}, x_1, t_2, \dots, t_j)}{(e^{x_1 t_2 \cdots t_j} - 1)^{\alpha_2 + \dots + \alpha_j + 1}} \right) \\ &= \sum_m \left(\prod_{j=2}^{k-1} \frac{f_{m,j}(e^{x_1 t_2, \dots, t_j}, x_1, t_2, \dots, t_j)}{(e^{x_1 t_2 \cdots t_j} - 1)^{\alpha_2 + \dots + \alpha_j + 1}} \right) \frac{\partial}{\partial x_k} \left(\prod_{j=k}^d \frac{f_{m,j}(e^{x_1 t_2, \dots, t_j}, x_1, t_2, \dots, t_j)}{(e^{x_1 t_2 \cdots t_j} - 1)^{\alpha_2 + \dots + \alpha_j + 1}} \right). \end{aligned} \quad (4.8)$$

Using the product rule to (4.8), we find

$$\begin{aligned} \frac{\partial}{\partial x_k} \left(\prod_{j=k}^d \frac{f_{m,j}(e^{x_1 t_2, \dots, t_j}, x_1, t_2, \dots, t_j)}{(e^{x_1 t_2 \dots t_j} - 1)^{\alpha_2 + \dots + \alpha_j + 1}} \right) \\ = \sum_{l=k}^d \left(\prod_{\substack{j=k \\ j \neq l}}^d \frac{f_{m,j}(e^{x_1 t_2, \dots, t_j}, x_1, t_2, \dots, t_j)}{(e^{x_1 t_2 \dots t_j} - 1)^{\alpha_2 + \dots + \alpha_j + 1}} \right) \frac{\partial}{\partial x_k} \left(\frac{f_{m,l}(e^{x_1 t_2, \dots, t_l}, x_1, t_2, \dots, t_l)}{(e^{x_1 t_2 \dots t_l} - 1)^{\alpha_2 + \dots + \alpha_l + 1}} \right). \end{aligned} \quad (4.9)$$

Using the quotient rule to (4.9), we get

$$\begin{aligned} \frac{\partial}{\partial x_k} \left(\frac{f_{m,l}(e^{x_1 t_2, \dots, t_l}, x_1, t_2, \dots, t_l)}{(e^{x_1 t_2 \dots t_l} - 1)^{\alpha_2 + \dots + \alpha_l + 1}} \right) \\ = \pm \frac{(e^{x_1 t_2 \dots t_l} - 1) \frac{\partial}{\partial x_k} f_{m,l} - (\alpha_1 + \dots + \alpha_l + 1) x_1 t_2 \dots t_{k-1} t_{k+1} \dots t_l f_{m,l}}{(e^{x_1 t_2 \dots t_l} - 1)^{\alpha_2 + \dots + \alpha_l + 2}}, \end{aligned} \quad (4.10)$$

where the choice of \pm depends on a_k . In the numerator of (4.10), the degree of $e^{x_1 t_2 \dots t_l}$ is less than or equal to $\alpha_2 + \dots + \alpha_l + 1$. Hence, Lemma 4 is valid for $(0, \alpha_2, \dots, \alpha_k + 1, \dots, \alpha_d)$. \square

Lemma 5. For each $\alpha = (0, \alpha_2, \dots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d$, $\partial^\alpha \psi_a(x_1, \dots, x_d)$ is bounded on $[0, \infty) \times [0, \frac{1}{2}] \times \dots \times [0, \frac{1}{2}]$.

Proof. By the Leibniz rule, we have

$$|\partial^\alpha \psi_a(x_1, \dots, x_d)| \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left| \partial^\beta \left(\prod_{j=2}^d (1 - x_j)^{v_j} \right) \right| \left| \partial^{\alpha - \beta} \left(\prod_{j=2}^d \frac{x_1 t_2 \dots t_j}{e^{x_1 t_2 \dots t_j} - 1} \right) \right|.$$

By Lemma 4, $\left| \partial^{\alpha - \beta} \left(\prod_{j=2}^d x_1 t_2 \dots t_j / (e^{x_1 t_2 \dots t_j} - 1) \right) \right|$ is bounded on $[0, \infty) \times [0, \frac{1}{2}] \times \dots \times [0, \frac{1}{2}]$. Hence what we have to prove is that $\left| \partial^\beta \prod_{j=2}^d (1 - x_j)^{v_j} \right|$ is bounded on $[0, \infty) \times [0, \frac{1}{2}] \times \dots \times [0, \frac{1}{2}]$. We find

$$\begin{aligned} \left| \partial^\beta \prod_{j=2}^d (1 - x_j)^{v_j} \right| &= \prod_{j=2}^d \left| \partial_{x_j}^{\beta_j} (1 - x_j)^{v_j} \right| \\ &= \prod_{j=2}^d \left| (v_j - \beta_j + 1)_{\beta_j} (1 - x_j)^{v_j - \beta_j} \right| \\ &\leq \prod_{j=2}^d \left| (v_j - \beta_j + 1)_{\beta_j} \right| \max \left\{ 1, \left| \frac{1}{2} \right|^{\Re(v_j - \beta_j)} \right\}, \end{aligned}$$

where $\beta = (\beta_1, \dots, \beta_d)$. \square

Lemma 6. Let $|a|, |b| \leq \frac{1}{2}$, $a \neq 0, b \neq 0$. Then for each $m, n \in \mathbb{Z}$, we have

$$\frac{(a)_n}{(b)_m} = \begin{cases} \frac{a}{b} \left(\frac{(n-1)!}{(m-1)!} + O(a) + O(b) \right) & (n \geq m \geq 1), \\ O(a) & (n \geq 1 > m), \\ (-1)^{m+n} \frac{(-m)!}{(-n)!} + O(a) + O(b) & (1 > n \geq m) \end{cases}$$

as $a, b \rightarrow 0$.

Proof. In the case $n \geq m \geq 1$, we have

$$\begin{aligned} \frac{(a)_n}{(b)_m} &= \frac{a}{b} \left(\frac{(a+1) \cdots (a+n-1)}{(b+1) \cdots (b+m-1)} \right) \\ &= \frac{a}{b} \left\{ \left((n-1)! + O(a) \right) \left(\frac{1}{(m-1)!} + O(b) \right) \right\} \\ &= \frac{a}{b} \left(\frac{(n-1)!}{(m-1)!} + O(a) + O(b) \right). \end{aligned}$$

In the case $n \geq 1 > m$, we have

$$\begin{aligned} \frac{(a)_n}{(b)_m} &= a(a+1) \cdots (a+n-1)(b-1) \cdots (b+m) \\ &\ll a. \end{aligned}$$

In the case $1 > n \geq m$, we have

$$\begin{aligned} \frac{(a)_n}{(b)_m} &= \frac{(b-1) \cdots (b+m)}{(a-1) \cdots (a+n)} \\ &= \{(-1)^m (-m)! + O(b)\} \{(-1)^n (-n)!^{-1} + O(a)\} \\ &= (-1)^{m+n} \frac{(-m)!}{(-n)!} + O(a) + O(b). \end{aligned}$$

□

5 Proof of Theorem 1

In this section, we prove Theorem 1.

By [[10], p1279, (7)], we have

$$\begin{aligned} &\Gamma(s_1) \cdots \Gamma(s_d) \zeta_d(s_1, \dots, s_d) \\ &= \int_0^1 \cdots \int_0^1 \int_0^\infty \prod_{j=1}^d x_j^{s_d(j)-d+j-2} \prod_{j=2}^d (1-x_j)^{s_{j-1}-1} \prod_{j=1}^d \frac{x_1 \cdots x_j}{e^{x_1 \cdots x_j} - 1} dx_1 \cdots dx_d. \end{aligned} \quad (5.1)$$

The right-hand side of (5.1) is divided into two terms,

$$\int_0^1 \cdots \int_0^1 \int_0^\infty = \int_0^1 \cdots \int_0^1 \int_0^1 + \int_0^1 \cdots \int_0^1 \int_1^\infty. \quad (5.2)$$

First we consider the first term of (5.2). By $x/(e^x - 1) = \sum_{m=0}^\infty (B_m/m!) x^m$ ($|x| < 2\pi$), we see that the first term is

$$\int_0^1 \cdots \int_0^1 \prod_{j=1}^d x_j^{s_d(j)-d+j-2} \prod_{j=2}^d (1-x_j)^{s_{j-1}-1} \prod_{j=1}^d \left(\sum_{k=0}^\infty \frac{B_k}{k!} (x_1 \cdots x_j)^k \right) dx_1 \cdots dx_d. \quad (5.3)$$

Further we divide the summation in (5.3) as

$$\begin{aligned} \prod_{j=1}^d \sum_{k=0}^{\infty} \frac{B_k}{k!} (x_1 \cdots x_j)^k &= \sum_{k=0}^{\infty} \sum_{p_1+\cdots+p_d=k} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} x_1^k x_2^{p_d(2)} \cdots x_d^{p_d(d)} \\ &= \sum_{k=0}^{n_1} \sum_{p_1+\cdots+p_d=k} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} x_1^k x_2^{p_d(2)} \cdots x_d^{p_d(d)} \\ &\quad + \sum_{k=n_1+1}^{\infty} \sum_{p_1+\cdots+p_d=k} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} x_1^k x_2^{p_d(2)} \cdots x_d^{p_d(d)}. \end{aligned} \quad (5.4)$$

The contribution of the first term of (5.4) is

$$\sum_{k=0}^{n_1} \sum_{p_1+\cdots+p_d=k} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} \frac{1}{s_d(1) - d + k} \prod_{j=2}^d B(s_d(j) - d + j + p_d(j) - 1, s_{j-1}). \quad (5.5)$$

This is the first term of (2.1). Changing the order of integration of the second term of (5.4), we have

$$\begin{aligned} \int_0^1 x_1^{s_d(1)-d+n_1} \int_0^1 \cdots \int_0^1 \left(\prod_{j=2}^d x_j^{s_d(j)-d+j-2} (1-x_j)^{s_{j-1}-1} \right) \\ \left(\sum_{k=n_1+1}^{\infty} \sum_{p_1+\cdots+p_d=k} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} x_1^{k-n_1-1} x_2^{p_d(2)} \cdots x_d^{p_d(d)} \right) dx_2 \cdots dx_d dx_1. \end{aligned} \quad (5.6)$$

Dividing the integral into $\int_0^{1/2}$ and $\int_{1/2}^1$ for x_2, \dots, x_d , we have

$$\int_0^1 \cdots \int_0^1 = \sum_{(a_2, \dots, a_d)} \int_{\frac{a_2}{2}}^{\frac{a_2+1}{2}} \cdots \int_{\frac{a_d}{2}}^{\frac{a_d+1}{2}},$$

where the notation $\sum_{(a_2, \dots, a_d)}$ is defined in the statement of Theorem 1. Changing variables, we find that (5.6) is

$$\begin{aligned} \int_0^1 x_1^{s_d(1)-d+n_1} \left\{ \sum_{(a_2, \dots, a_d)} \int_0^{\frac{1}{2}} \cdots \int_0^{\frac{1}{2}} \left(\prod_{j=2}^d x_j^{u_j} (1-x_j)^{v_j} \right) \right. \\ \left. \left(\sum_{k=n_1+1}^{\infty} \sum_{p_1+\cdots+p_d=k} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} x_1^{k-n_1-1} t_2^{p_d(2)} \cdots t_d^{p_d(d)} \right) dx_2 \cdots dx_d \right\} dx_1. \end{aligned}$$

By Lemma 2, Lemma 3 and the definition of $\varphi_a(x_1, \dots, x_d)$, we find that the above is

$$\int_0^1 x_1^{s_d(1)-d+n_1} F_{\varphi_a}(x_1) dx_1. \quad (5.7)$$

By (5.5) and (5.7), we see that the first term of (5.2) is

$$\begin{aligned} \sum_{k=0}^{n_1} \sum_{p_1+\cdots+p_d=k} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} \frac{1}{s_d(1) - d + k} \prod_{j=2}^d B(s_d(j) - d + j + p_d(j) - 1, s_{j-1}) \\ + \int_0^1 x_1^{s_d(1)-d+n_1} F_{\varphi_a}(x_1) dx_1. \end{aligned} \quad (5.8)$$

Next, we consider the second term of (5.2). Similarly to the deformation of (5.6), we have

$$\int_1^\infty \frac{x_1^{s_d(1)-d}}{e^{x_1}-1} \left(\sum_{(a_2, \dots, a_d)} \int_0^{\frac{1}{2}} \cdots \int_0^{\frac{1}{2}} \prod_{j=2}^d x_j^{u_j} (1-x_j)^{v_j} \frac{x_1 t_2 \cdots t_j}{e^{x_1 t_2 \cdots t_j}-1} dx_2 \cdots dx_d \right) dx_1.$$

Using Lemma 2, we find that the above is

$$\int_1^\infty \frac{x_1^{s_d(1)-d}}{e^{x_1}-1} F_{\psi_a}(x_1) dx_1. \quad (5.9)$$

By (5.8) and (5.9), we obtain (2.1).

Now let us consider when (2.1) is holomorphic. The first term is holomorphic when

$$\begin{aligned} s_d(1) &\neq d, d-1, \dots, d-n_1, \\ s_d(j) &\neq d-j+1, d-j, d-j-1, \dots & (j=2, \dots, d), \\ s_j &\neq 0, -1, -2, \dots & (j=1, \dots, d-1). \end{aligned}$$

By Lemma 3, the second term is holomorphic when

$$\begin{aligned} s_d(j) &\neq d-j+1, d-j, \dots, d-j+1-n_j & (j=2, \dots, d), \\ s_j &\neq 0, -1, \dots, -n_j & (j=2, \dots, d-1), \\ \Re(s_d(j)) &> d-j-n_j & (j=1, \dots, d), \\ \Re(s_{j-1}) &> -n_j-1 & (j=2, \dots, d). \end{aligned}$$

By Lemma 5, the third term is holomorphic when

$$\begin{aligned} s_d(j) &\neq d-j+1, d-j, \dots, d-j+1-n_j & (j=2, \dots, d), \\ s_j &\neq 0, -1, \dots, -n_j & (j=2, \dots, d-1), \\ \Re(s_d(j)) &> d-j-n_j & (j=2, \dots, d), \\ \Re(s_{j-1}) &> -n_j-1 & (j=2, \dots, d). \end{aligned}$$

Hence, we obtain Theorem 1.

6 Proof of Theorem 2

In this section, we prove Theorem 2. If $d=1$, $\zeta_1(s_1)$ is Riemann zeta function. Hence, Theorem 2 is clear. So we prove Theorem 2 in the case $d>1$. Suppose that m_j, ε_j ($j=1, \dots, d$) and M are defined in the statement of Theorem 2. We use (2.1) with $s_j = -m_j + \varepsilon_j$ ($j=1, \dots, d$) and $n_1 = \dots = n_d = M+d$.

First, we estimate the second term and the third term. When $(\varepsilon_1, \dots, \varepsilon_d) \rightarrow (0, \dots, 0)$, these terms are bounded except $(u_{\sigma(j)}+1)_{n_{\sigma(j)}+1}^{-1}$ and $(u_{\sigma(j)}+1)_{k_{\sigma(j)}+1}^{-1}$. Hence, we have

$$\int_0^1 x_1^{s_d(1)-d+n_1} F_{\varphi_a}(x_1) dx_1 + \int_1^\infty \frac{x_1^{s_d(1)-d}}{e^{x_1}-1} F_{\psi_a}(x_1) dx_1 = \sum_{(a_2, \dots, a_d)} O\left(\prod_{j=2}^d w_j^{-1}\right)$$

where

$$w_j := \begin{cases} \varepsilon_d(j) & (a_j = 0) \\ \varepsilon_{j-1} & (a_j = 1). \end{cases}$$

On the other hand, using $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we can estimate

$$\begin{aligned} \frac{1}{\Gamma(s_1) \cdots \Gamma(s_d)} &\ll \sin(\pi s_1) \cdots \sin(\pi s_d) \\ &\ll \sin(\pi \varepsilon_1) \cdots \sin(\pi \varepsilon_d) \\ &\ll \varepsilon_1 \cdots \varepsilon_d. \end{aligned}$$

Then, we have

$$\begin{aligned} &\frac{1}{\Gamma(s_1) \cdots \Gamma(s_d)} \left(\int_0^1 x_1^{s_d(1)-d+n_1} F_{\varphi_a}(x_1) dx_1 + \int_1^\infty \frac{x_1^{s_d(1)-d}}{e^{x_1}-1} F_{\psi_a}(x_1) dx_1 \right) \\ &= \sum_{(a_1, \dots, a_d)} O \left(\left(\prod_{j=2}^d \frac{\varepsilon_{j-1}}{w_j} \right) \varepsilon_d \right) \\ &= \sum_{(a_1, \dots, a_d)} O \left(\left(\prod_{\substack{j=2 \\ a_j=0}}^d \frac{\varepsilon_{j-1}}{\varepsilon_d(j)} \right) \varepsilon_d \right). \end{aligned}$$

Since $\varepsilon_k/\varepsilon_d(j) \ll 1$ ($j = 1, \dots, d$, $k = j, \dots, d$), we obtain

$$\frac{1}{\Gamma(s_1) \cdots \Gamma(s_d)} \left(\int_0^1 x_1^{s_d(1)-d+n_1} F_{\varphi_a}(x_1) dx_1 + \int_1^\infty \frac{x_1^{s_d(1)-d}}{e^{x_1}-1} F_{\psi_a}(x_1) dx_1 \right) = \sum_{j=1}^d O(\varepsilon_j). \quad (6.1)$$

Next, we estimate the first term of (2.1). First, we estimate the factors containing gamma functions and beta functions as

$$\begin{aligned} &\frac{1}{\Gamma(s_1) \cdots \Gamma(s_d)} \prod_{j=2}^d B(s_d(j) - d + j + p_d(j) - 1, s_{j-1}) \\ &= \frac{1}{\Gamma(s_d)} \prod_{j=2}^d \frac{\Gamma(s_d(j) - d + j + p_d(j) - 1)}{\Gamma(s_d(j-1) - d + j + p_d(j) - 1)} \\ &= \frac{1}{(\varepsilon_d)_{-m_d} \Gamma(\varepsilon_d(1))} \prod_{j=2}^d \frac{(\varepsilon_d(j))_{s_d(j)-d+j+p_d(j)-1}}{(\varepsilon_d(j-1))_{s_d(j-1)-d+j+p_d(j)-1}}. \end{aligned} \quad (6.2)$$

By Lemma 6, we have

$$\begin{aligned} \frac{1}{(\varepsilon_d)_{-m_d} \Gamma(\varepsilon_d(1))} &= ((-1)^{m_d} m_d! + O(\varepsilon_d)) \left(\frac{\sin(\pi \varepsilon_d(1))}{\pi} \Gamma(1 - \varepsilon_d(1)) \right) \\ &= (-1)^{m_d} m_d! \varepsilon_d(1) + O(\varepsilon_d(1)^2) + O(\varepsilon_d(1) \varepsilon_d) \end{aligned}$$

and

$$\begin{aligned} &\prod_{j=2}^d \frac{(\varepsilon_d(j))_{s_d(j)-d+j+p_d(j)-1}}{(\varepsilon_d(j-1))_{s_d(j-1)-d+j+p_d(j)-1}} \\ &= \prod_{j=2}^d \left(h(-m_d(j) - d + j + p_d(j) - 1, -m_d(j-1) - d + j + p_d(j) - 1) \frac{[\varepsilon_d(j)]_{-m_d(j)-d+j+p_d(j)-1}}{[\varepsilon_d(j-1)]_{-m_d(j-1)-d+j+p_d(j)-1}} \right) \\ &\quad + \sum_{j=2}^d \left\{ O \left(\frac{\varepsilon_d(j)}{\varepsilon_d(j-1)} \varepsilon_d(j) \right) + O(\varepsilon_d(j-1)) + O(\varepsilon_d(j)) \right\}, \end{aligned}$$

hence, we find (6.2) is

$$\begin{aligned} & \frac{1}{\Gamma(s_1) \cdots \Gamma(s_d)} \prod_{j=2}^d B(s_d(j) - d + j + p_d(j) - 1, s_{j-1}) = (-1)^{m_d} m_d! \varepsilon_d(1) \times \\ & \times \prod_{j=2}^d \left(h(-m_d(j) - d + j + p_d(j) - 1, -m_d(j-1) - d + j + p_d(j) - 1) \frac{[\varepsilon_d(j)]_{-m_d(j)-d+j+p_d(j)-1}}{[\varepsilon_d(j-1)]_{-m_d(j-1)-d+j+p_d(j)-1}} \right) \\ & + \sum_{j=1}^d O(\varepsilon_j \varepsilon_d(1)). \quad (6.3) \end{aligned}$$

Using (6.3), we can estimate the first term of (2.1),

$$\begin{aligned} & \sum_{k=0}^{n_1} \sum_{p_1+\cdots+p_d=k} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} \frac{1}{s_d(1) - d + k} \left\{ (-1)^{m_d} m_d! \varepsilon_d(1) \times \right. \\ & \times \prod_{j=2}^d \left(h(-m_d(j) - d + j + p_d(j) - 1, -m_d(j-1) - d + j + p_d(j) - 1) \frac{[\varepsilon_d(j)]_{-m_d(j)-d+j+p_d(j)-1}}{[\varepsilon_d(j-1)]_{-m_d(j-1)-d+j+p_d(j)-1}} \right) \\ & \left. + \sum_{j=1}^d O(\varepsilon_d(j) \varepsilon_d(1)) \right\}. \end{aligned}$$

Using

$$\frac{1}{s_d(1) - d + k} = \begin{cases} O(1) & (k < n_1 = M + d) \\ \varepsilon_d(1)^{-1} & (k = n_1 = M + d), \end{cases}$$

we have

$$\begin{aligned} & (-1)^{m_d} m_d! \sum_{p_1+\cdots+p_d=d+M} \frac{B_{p_1} \cdots B_{p_d}}{p_1! \cdots p_d!} \prod_{j=2}^d h(-m_d(j) - d + j + p_d(j) - 1, -m_d(j-1) - d + j + p_d(j) - 1) \\ & \times \frac{[\varepsilon_d(j)]_{-m_d(j)-d+j+p_d(j)-1}}{[\varepsilon_d(j-1)]_{-m_d(j-1)-d+j+p_d(j)-1}} + \sum_{j=1}^d O(\varepsilon_j). \quad (6.4) \end{aligned}$$

From (6.1) and (6.4), we obtain Theorem 2.

References

- [1] S. Akiyama, S. Egami and Y. Tanigawa, *Analytic continuation of multiple zeta-functions and their values at non-positive integers*, Acta Arith, **98** (2001), 107-116.
- [2] S. Akiyama, Y. Tanigawa, *Multiple zeta values at non-positive integers*, Ramanujan J. **5** (2001), 327-351.
- [3] I. M. Gelfand and G. E. Shilov, *Generalized Function* vol. I, Academic Press, New York and London (1964).
- [4] K. Kamano, *The multiple Hurwitz zeta function and a generalization of Lerch's formula*, Tokyo J. Math. **29** (2006), 61-73.

- [5] Y. Komori, *An integral representation of multiple Hurwitz-Lerch zeta functions and generalized multiple bernoulli numbers*, Quart. J. Math. (Oxford) (2009), 1-60.
- [6] K. Matsumoto, *On analytic continuation of various multiple zeta-functions*, Number Theory for the Millenium (Urbana, 2000), Vol. II, M. A. Bennett et. al. (eds.), A. K. Peters, Natick, MA, 2002, pp. 417-440.
- [7] K. Matsumoto, *The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions I*, J. Number Theory **101** (2003), 223-243.
- [8] Y. Sasaki, *Multiple zeta values for coordinatewise limits at non-positive integers*, Acta Arith. **136** (2009), 299-317.
- [9] Y. Sasaki, *Some formulas of multiple zeta values for coordinate-wise limits at non-positive integers*, in "New Directions in Value-Distribution Theory of Zeta and L -Functions" (Wuerzburg Conference, Oct 6-10, 2008), R. Steuding & J. Steuding (eds.), Shaker Verlag, 2009, pp.317-325.
- [10] J. Zhao, *Analytic continuation of multiple zeta functions*, Proc. Amer. Math. Soc. **128** (2000), 1275-1283.

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